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Extremal Homogeneous Polynomials on Real Normed Spaces

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If P is a continuous m-homogeneous polynomial on a real normed space and $\check{P}$ is the associated symmetric m-linear form, the ratio $\|\check{P}\|/\|P\|$ always lies between 1 and $m^m/m!$. We show that, as in the complex case investigated by Sarantopoulos (1987, *Proc. Amer. Math. Soc.* **99**, 340–346), there are P's for which $\|\check{P}\|/\|P\| = m^m/m!$ and for which $\check{P}$ achieves norm if and only if the normed space contains an isometric copy of ℓ_1^m. However, unlike the complex case, we find a plentiful supply of such polynomials provided $m \geqslant 4$. © 1999 Academic Press

1. INTRODUCTION AND NOTATION

Let E be a vector space over K, where $K = \mathbb{R}$ or $\mathbb{C}$. A mapping $P: E \to K$ is said to be an *m-homogeneous polynomial on* E if $P(s\mathbf{x} + t\mathbf{y})$ is an m-homogeneous polynomial (in the algebraic sense) in $s, t \in K$ for arbitrary fixed $\mathbf{x}, \mathbf{y}$ in E. It follows easily that, for any $k \geqslant 2$, $P(t_1\mathbf{x}_1 + \cdots + t_k\mathbf{x}_k)$ is an m-homogeneous polynomial (in the algebraic sense) in $t_1, ..., t_k \in K$ for arbitrary fixed $\mathbf{x}_1, ..., \mathbf{x}_k \in E$. Consequently, for a finite-dimensional vector

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space E, this abstract definition of an m-homogeneous polynomial coincides with the usual algebraic definition.

As shown in Hörmander [3, Lemma 1] (see also [2]), to each m-homogeneous polynomial P on E there corresponds a unique symmetric m-linear form $\check{P}$ on E such that $\check{P}(\mathbf{x}, ..., \mathbf{x}) = P(\mathbf{x})$. The map $\check{P}$ has a variety of common names, including *the* mth *polar of* P, *the polarized form of* P, and *the blossom of* P. It is defined by

$$\check{P}(\mathbf{x}_1, ..., \mathbf{x}_m) := \frac{1}{m!} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} P(t_1 \mathbf{x}_1 + \cdots + t_m \mathbf{x}_m).$$

From this it is easy to see that $\check{P}(\mathbf{x}_1, ..., \mathbf{x}_m)$ is $1/m!$ times the coefficient of $t_1 \cdots t_m$ in the expansion of $P(t_1 \mathbf{x}_1 + \cdots + t_m \mathbf{x}_m)$ as a polynomial in $t_1, ..., t_m \in K$. In particular, if $E = K^n$ the definition of $\check{P}$ agrees with another standard one involving derivatives, namely

$$\check{P}(\mathbf{x}_1, ..., \mathbf{x}_m) := \frac{1}{m!} \left(\prod_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \frac{\partial}{\partial x_j} \right) P(\mathbf{x}),$$

where $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{x}_i = (x_{i1}, ..., x_{in})$ $(1 \leqslant i \leqslant m)$. (Note that the right-hand side is independent of $\mathbf{x}$.) In fact, it is easy to check that $\check{P}$ is symmetric and linear in each variable separately. Euler's identity for homogeneous polynomials can be used to show that $\check{P}(\mathbf{x}, ..., \mathbf{x}) = P(\mathbf{x})$.

If E is a normed space, then P is continuous if and only if $\check{P}$ is continuous. We define multilinear and polynomial norms by

$$\|\check{P}\| = \sup\{|\check{P}(\mathbf{x}_1, ..., \mathbf{x}_m)|: \|\mathbf{x}_i\| \leqslant 1, 1 \leqslant i \leqslant m\};$$
$$\|P\| = \sup\{|P(\mathbf{x})|: \|\mathbf{x}\| \leqslant 1\}.$$

These norms are equivalent, and Martin [5] proved that

$$\|P\| \leqslant \|\check{P}\| \leqslant \frac{m^m}{m!} \|P\|$$

for every continuous m-homogeneous polynomial P on E. A standard reference is [1, p. 5].

We recall an example, due to Nachbin, which shows that equality can be achieved on the right-hand side. Let ℓ_1^m be the space $\mathbb{R}^m$ with the norm

$$\|\mathbf{x}\| := |x_1| + \cdots + |x_m|,$$

where $\mathbf{x} = (x_1, ..., x_m)$. Define the m-homogeneous Nachbin polynomial N on ℓ_1^m by

$$N(\mathbf{x}) := x_1 \cdots x_m.$$

Then

$$\check{N}(\mathbf{x}_1, ..., \mathbf{x}_m) = \frac{1}{m!} \sum_{\sigma \in S_m} x_{\sigma(1)\,1} \cdots x_{\sigma(m)\,m},$$

where $\mathbf{x}_i = (x_{i1}, ..., x_{im})$ for $1 \leqslant i \leqslant m$ and where S_m is the set of permutations of the first m natural numbers. It is not difficult to see [1, p. 6] that

$$\|\check{N}\| = \frac{1}{m!} \qquad \text{and} \qquad \|N\| = \frac{1}{m^m}.$$

The object of this paper is to study which m-homogeneous polynomials share the extremal properties of N.

Definition. If E is a real normed space and P is a continuous m-homogeneous polynomial on E, we say that P is *extremal* if

(i) $\|\check{P}\| = (m^m/m!) \cdot \|P\|$ and

(ii) there exist $\mathbf{x}_1, ..., \mathbf{x}_m$ in the unit sphere of E with $\|\check{P}\| = \check{P}(\mathbf{x}_1, ..., \mathbf{x}_m)$.

Note that N automatically has property (ii). The unit sphere of ℓ_1^m is compact.

Sarantopoulos [7] investigated extremal polynomials on *complex* normed spaces. A useful tool for him was the *complex* normed space version of the following reduction lemma. The proof for real normed spaces requires only slight changes to Sarantopoulos' argument.

Theorem 1 (Reduction Lemma). *Suppose P is an extremal m-homogeneous polynomial on a real normed space E and*

$$\|\check{P}\| = \check{P}(\mathbf{x}_1, ..., \mathbf{x}_m),$$

where $\mathbf{x}_1, ..., \mathbf{x}_m$ are in the unit sphere of E. Then for every m-tuple $(a_1, ..., a_m)$ of real numbers we have

$$\|a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m\| = |a_1| + \cdots + |a_m|.$$

(*Thus* $\operatorname{span}\{\mathbf{x}_1, ..., \mathbf{x}_m\} \subseteq E$ *is isometrically isomorphic to ℓ_1^m, and the isomorphism maps $\{\mathbf{x}_1, ..., \mathbf{x}_m\}$ to the standard unit vector basis of ℓ_1^m.*)

Sarantopoulos showed that if P is an extremal m-homogeneous polynomial on a *complex* normed space E, then the restriction of P to the isometric copy of ℓ_1^m found in the complex version of Theorem 1 is just a multiple of Nachbin's polynomial of degree m. In particular, the only extremal m-homogeneous polynomials on *complex* ℓ_1^m are the multiples of Nachbin's polynomial. However, although Theorem 1 is valid for real or complex normed spaces, we shall show that, when $m \geqslant 4$, the multiples of Nachbin's polynomials are not the only extremal polynomials on real ℓ_1^m.

To give an idea of why there is a difference between the real and complex cases, we consider the Bochnak complexification $\tilde{E} = E \hat{\otimes} \ell_2^2$ of a real normed space E. (See [4] or [6] for an extended discussion of complexifications of real normed spaces.) The norm on $\tilde{E}$ is given by

$$\|t\|_{\tilde{E}} := \inf \{ \Sigma_k \|x_k\|_E \|y_k\|_{\ell_2^2} : t = \Sigma_k x_k \otimes y_k \}.$$

In our context, it is important to note that the Bochnak complexification of any real $L_1(\mu)$ is the corresponding complex $L_1(\mu)$.

Each continuous m-homogeneous polynomial P on E has a unique extension $\tilde{P}$ which is a continuous m-homogeneous polynomial on $\tilde{E}$. Moreover, if we write $L = \check{P}$, then $\|\tilde{L}\| = \|L\|$, where $\tilde{L}$ is the unique extension of L, but in general we can only say that $\|\tilde{P}\| \geqslant \|P\|$.

Now, if $\|\tilde{P}\|$ is extremal, then

$$\|L\| = \|\tilde{L}\| = \frac{m^m}{m!} \|\tilde{P}\| \geqslant \frac{m^m}{m!} \|P\|$$

and so P is extremal on E. The converse is not generally true. However, the converse is true when $m = 2$, because $\|P\| = \|\tilde{P}\|$ for any 2-homogeneous polynomial P on any real normed space. (See the comments after Proposition 20 in [6].) It follows from all this that the only extremal 2-homogeneous polynomials on either real or complex ℓ_1^2 are the multiples of Nachbin's polynomial of degree 2.

In this paper we show, by completely different methods, that it is also true that the only extremal 3-homogeneous polynomials on ℓ_1^3 are the multiples of Nachbin's polynomial of degree 3, but that this analogy with the complex case breaks down for extremal m-homogeneous polynomials on real ℓ_1^m for every $m \geqslant 4$. In this case, we show that the supply of extremal m-homogeneous polynomials is much larger than before, and *suitable* perturbations of Nachbin's example remain extremal.

2. NORMALIZED EXTREMAL POLYNOMIALS

Clearly, any multiple of an extremal m-homogeneous polynomial P on a real normed space E is still extremal, and the importance of the Nachbin polynomials in the complex case prompts us to assume from now on that

$\|\check{P}\| = 1/m!$ and $\|P\| = m^{-m}$. In view of the Reduction Lemma, we shall further restrict attention to the situation where P is an m-homogeneous polynomial on ℓ_1^m. This enables us to compute the norm of the associated symmetric m-linear form with ease:

$$\|\check{P}\| = \max\{|\check{P}(\mathbf{e}_{k_1}, ..., \mathbf{e}_{k_m})|: 1 \leqslant k_1, ..., k_m \leqslant m\} \tag{$*$}$$

where $\{\mathbf{e}_1, ..., \mathbf{e}_m\}$ is the standard unit vector basis of ℓ_1^m. Referring one more time to the Reduction Lemma, the fact that $1/m! = \|\check{P}\| = \check{P}(\mathbf{e}_1, ..., \mathbf{e}_m)$ constrains P to have the form

$$P(\mathbf{x}) = x_1 \cdots x_m + \sum a_{k_1 \ldots k_m} x_1^{k_1} \cdots x_m^{k_m}$$

where $\mathbf{x} = (x_1, ..., x_m)$ and the summation is over all $(k_1, ..., k_m)$ with at least one k_i greater than 1. In view of all this, a definition is called for.

Definition. An m-homogeneous polynomial P on ℓ_1^m is *normalized extremal* if

(i) $P(\mathbf{x}) = x_1 \cdots x_m + E(\mathbf{x})$, where $\mathbf{x} = (x_1, ..., x_m)$ and the terms in $E(\mathbf{x})$ have at least one variable raised to a power greater than 1,

(ii) $\|P\| = m^{-m}$,

(iii) $\|\check{P}\| = 1/m!$.

We write $\mathscr{E}_m$ for the set of all normalized extremal m-homogeneous polynomials on ℓ_1^m.

It will be important for us to have detailed knowledge of the behaviour of normalized extremal m-homogeneous polynomials at the points where they attain norm.

Theorem 2. *Let $P \in \mathscr{E}_m$ and let $\boldsymbol{\varepsilon} = (\varepsilon_1, ..., \varepsilon_m)$ with each $\varepsilon_i = \pm 1$. Then*

(a) $P(\boldsymbol{\varepsilon}/m) = \varepsilon_1 \cdots \varepsilon_m / m^m$,

(b) $P(\boldsymbol{\varepsilon}/m) = (\varepsilon_i/m) \cdot \partial P/\partial x_i\, (\boldsymbol{\varepsilon}/m)$ *for each* $1 \leqslant i \leqslant m$, *and*

(c) $\partial E/\partial x_i(\boldsymbol{\varepsilon}/m) = 0$ *for each* $1 \leqslant i \leqslant m$.

Proof. (a) Since $P \in \mathscr{E}_m$, it follows from the classical polarization formula (see [1, p. 4]) that

$$\frac{1}{m!} = \check{P}(\mathbf{e}_1, ..., \mathbf{e}_m) = \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_m P\left(\sum_{i=1}^m \varepsilon_i \mathbf{e}_i\right) \leqslant \frac{m^m}{2^m m!} \sum_{\varepsilon_i = \pm 1} |P(\boldsymbol{\varepsilon}/m)|$$

$$\leqslant \frac{m^m}{2^m m!} \|P\| \sum_{\varepsilon_i = \pm 1} \|\boldsymbol{\varepsilon}/m\|^m = \frac{1}{m!}.$$

As all the above inequalities are equalities it follows that $P(\boldsymbol{\varepsilon}/m) = \varepsilon_1 \cdots \varepsilon_m/m^m$, as required for (a).

Since P achieves its norm at $\boldsymbol{\varepsilon}/m$ it has a relative extremum at $\boldsymbol{\varepsilon}/m$ when it is subjected to the constraint $g(x_1, ..., x_m) := \varepsilon_1 x_1 + \cdots + \varepsilon_m x_m - 1 = 0$. The Lagrange multiplier method now tells us that for some real λ we have

$$\frac{\partial P}{\partial x_i}(\boldsymbol{\varepsilon}/m) = \lambda \varepsilon_i \qquad \text{for each} \quad 1 \leqslant i \leqslant m.$$

When we combine this with Euler's identity $mP = \sum_{i=1}^{m} x_i \, \partial P/\partial x_i$, we obtain that $\lambda = mP(\boldsymbol{\varepsilon}/m)$. It follows that

$$P(\boldsymbol{\varepsilon}/m) = (\varepsilon_i/m)\, \lambda \varepsilon_i = (\varepsilon_i/m) \frac{\partial P}{\partial x_i}(\boldsymbol{\varepsilon}/m) \qquad \text{for each} \quad 1 \leqslant i \leqslant m,$$

which proves (b).

Finally, observe that for each $1 \leqslant i \leqslant m$, $x_i \, \partial P/\partial x_i = x_1 \cdots x_m + x_i \, \partial E/\partial x_i$, and so when $\mathbf{x} = \boldsymbol{\varepsilon}/m$, parts (a) and (b) combined lead to (c).

Remark. It is part of the definition of an extremal m-homogeneous polynomial P on a normed space E that $\check{P}$ achieves its norm. In fact, P also achieves its norm, since if $\|\check{P}\| = \check{P}(\mathbf{x}_1, ..., \mathbf{x}_m)$, where $\mathbf{x}_1, ..., \mathbf{x}_m$ are in the unit sphere of E, then, working as in the proof of Theorem 2(a) it follows that $|P((\varepsilon_1 \mathbf{x}_1 + \cdots + \varepsilon_m \mathbf{x}_m)/m)| = \|P\| = m^{-m}$, with each $\varepsilon_i = \pm 1$. Surprisingly, in [8] a norm attaining m-homogeneous polynomial P satisfying $\|\check{P}\| = 1/m!$ and $\|P\| = m^{-m}$ has been constructed on the space $E := (\bigoplus_{n=m}^{\infty} E_n)_{\ell_1}$, where each E_n is a copy of ℓ_1^n, distorted in such a way that $\check{P}$ does not attain its norm.

We are grateful to the referee for showing us how to prove the next result, which gives considerable information about the structure of $\mathscr{E}_m$.

THEOREM 3. *If $P \in \mathscr{E}_m$, then $P(\mathbf{x}) = x_1 \cdots x_m + E(\mathbf{x})$, where $E(\mathbf{x})$ is in the ideal generated by*

$$\{(x_1^2 - x_i^2)(x_1^2 - x_j^2) \colon 1 < i \leqslant j \leqslant m\}.$$

Proof. Parts (a) and (c) of Theorem 2 tell us that E and all its first order partial derivatives have value 0 at each of the 2^m points $\boldsymbol{\varepsilon} = (\varepsilon_1, ..., \varepsilon_m)$. We will use this information to obtain the desired structure of E.

The first step is to notice that we can write

$$E(\mathbf{x}) = \sum_{\delta} x_1^{\delta_1} \cdots x_m^{\delta_m} E_\delta(x_1^2, ..., x_m^2),$$

where the summation is over $2^{m-1}-1$ possible choices of $\delta \in \{0, 1\}^m$. (Note that $\delta_1 + \cdots + \delta_m$ must have the same parity as m, and so once $\delta_1, ..., \delta_{m-1}$ are chosen, δ_m is determined. The choice $\delta = (1, ..., 1)$ is not permitted.) Notice that each E_δ is a homogeneous polynomial of positive degree.

Next, observe that if $r_1, ..., r_m$ denote the first m Rademacher functions, then for every $0 \leqslant t \leqslant 1$ we have

$$0 = E(r_1(t), ..., r_m(t)) = \sum_\delta r_1(t)^{\delta_1} \cdots r_m(t)^{\delta_m} E_\delta(1, ..., 1).$$

As distinct Rademacher products are orthonormal, it follows that

$$E_\delta(1, ..., 1) = 0$$

for every permissible $\delta \in \{0, 1\}^m$.

Now, a quick computation shows that for each $1 \leqslant i \leqslant m$ we have

$$\frac{\partial E}{\partial x_i}(\mathbf{x}) = \sum_\delta \frac{\partial}{\partial x_i}(x_1^{\delta_1} \cdots x_m^{\delta_m}) E_\delta(x_1^2, ..., x_m^2) + 2x_i \sum_\delta x_1^{\delta_1} \cdots x_m^{\delta_m} \frac{\partial E_\delta}{\partial x_i}(x_1^2, ..., x_m^2).$$

Consequently, for every $0 \leqslant t \leqslant 1$,

$$\frac{\partial E}{\partial x_i}(r_1(t), ..., r_m(t)) = 2r_i(t) \sum_\delta r_1(t)^{\delta_1} \cdots r_m(t)^{\delta_m} \frac{\partial E_\delta}{\partial x_i}(1, ..., 1).$$

The same argument as before now gives

$$\frac{\partial E_\delta}{\partial x_i}(1, ..., 1) = 0$$

for every $1 \leqslant i \leqslant m$ and every permissible $\delta \in \{0, 1\}^m$.

The result is now close. For each permissible δ define

$$F_\delta(y_1, ..., y_m) := E_\delta(y_1, y_1 + y_2, ..., y_1 + y_m).$$

By what we have just shown, F_δ is a homogeneous polynomial of positive degree, d say, which vanishes along with all its first partial derivatives at the point $(1, 0, ..., 0)$. Consequently, it cannot contain terms of the type y_1^d or $y_1^{d-1} y_i$ for any $2 \leqslant i \leqslant m$. This allows us to say that F_δ is in the ideal generated by $\{y_i y_j : 2 \leqslant i \leqslant j \leqslant m\}$. Translating, we find that each $E_\delta(x_1^2, ..., x_m^2)$, and hence $E(\mathbf{x})$, is in the announced ideal.

Remark. A close look at the proof of Theorem 3 reveals that $E_\delta(y_1, ..., y_m)$ cannot have degree 1, and from this it is clear that the only

3-homogeneous normalized extremal on real ℓ_1^3 is Nachbin's polynomial. We give a different proof of this in the next section.

One further structural result on $\mathscr{E}_m$ is quick to obtain.

PROPOSITION 4. *$\mathscr{E}_m$ is a convex set.*

Proof. Let P, Q be elements of $\mathscr{E}_m$ and let $0 \leqslant t \leqslant 1$. Evidently $R := tP + (1-t)\,Q$ satisfies (i), and so, thanks to $(*)$, $\|\check{R}\| \geqslant 1/m!$. Since $\|\check{R}\| \leqslant t\,\|\check{P}\| + (1-t)\|\check{Q}\|$, we also have $\|\check{R}\| \leqslant 1/m!$ and hence R satisfies (iii). Next, the triangle inequality gives $\|R\| \leqslant m^{-m}$, whereas the fact that $\|\check{R}\|/\|R\| \leqslant m^m/m!$ forces $\|R\| \geqslant m^{-m}$. Hence R satisfies (ii) and we are done.

3. THE CASES $\mathscr{E}_2$ AND $\mathscr{E}_3$

For small values of m, $\mathscr{E}_m$ is a very small set—just as in the complex case.

PROPOSITION 5. *The Nachbin polynomial $N(\mathbf{x}) = x_1 x_2$ is the only element of $\mathscr{E}_2$.*

Proof. Let $P \in \mathscr{E}_2$. Then $P(\mathbf{x}) = x_1 x_2 + E(\mathbf{x})$, where $E(\mathbf{x}) = a x_1^2 + b x_2^2$. By Theorem 2(c), $0 = \partial E/\partial x_1 = a$ at the point $(\frac{1}{2}, \frac{1}{2})$. Similarly $b = 0$ and we are done.

$\mathscr{E}_3$ is also a singleton, but to prove this it is handy to appeal to a simple lemma whose proof is a direct consequence of the definition of $\mathscr{E}_m$.

LEMMA 6. *Let $P \in \mathscr{E}_m$ and for each $1 \leqslant i \leqslant m$ define P_i by*

$$P_i(\mathbf{x}) := -P(x_1, ..., x_{i-1}, -x_i, x_{i+1}, ..., x_m).$$

Then $P_i \in \mathscr{E}_m$.

PROPOSITION 7. *The Nachbin polynomial $N(\mathbf{x}) = x_1 x_2 x_3$ is the only element of $\mathscr{E}_3$.*

Proof. Let $P \in \mathscr{E}_3$. Then

$$P(\mathbf{x}) = x_1 x_2 x_3 + \sum_{1 \leqslant i \leqslant 3} a_i x_i^3 + \sum_{1 \leqslant i \neq j \leqslant 3} b_{ij} x_i x_j^2.$$

By applying Lemma 6 and the convexity of $\mathscr{E}_3$ we see that $P^{(1)} := \frac{1}{2}(P + P_1)$ is also in $\mathscr{E}_3$. But $P^{(1)}$ is derived from P by deleting all terms with even powers of x_1, and so

$$P^{(1)}(x) = x_1 x_2 x_3 + a_1 x_1^3 + b_{12} x_1 x_2^2 + b_{13} x_1 x_3^2 := x_1 x_2 x_3 + E^{(1)}(\mathbf{x}).$$

Now, using Theorem 2(c), compute successively $\partial E^{(1)}/\partial x_3$, $\partial E^{(1)}/\partial x_2$, $\partial E^{(1)}/\partial x_1$, at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to find that $b_{13}=b_{12}=a_1=0$.

A similar argument using $P^{(2)}:=\frac{1}{2}(P+P_2)$ and $P^{(3)}:=\frac{1}{2}(P+P_3)$ shows that all a_i's and b_{ij}'s are zero, and so $P(\mathbf{x})=x_1x_2x_3$ as required.

4. THE CASES $\mathscr{E}_m$ FOR $m \geqslant 4$

The plot thickens when $m=4$.

THEOREM 8. *All $P \in \mathscr{E}_4$ have the form*

$$P(\mathbf{x})=x_1x_2x_3x_4+\sum_{1\leqslant i<j\leqslant 4} \gamma_{ij}(x_i^2-x_j^2)^2.$$

Proof. If $P \in \mathscr{E}_4$ then

$$P(\mathbf{x})=x_1x_2x_3x_4+\sum_{1\leqslant i\leqslant 4} a_i x_i^4+\sum_{1\leqslant i\neq j\leqslant 4} b_{ij}x_ix_j^3$$

$$+\sum_{1\leqslant i<j\leqslant 4} c_{ij}x_i^2x_j^2+\sum d_{ijk}x_i^2x_jx_k,$$

where the final sum is taken over all triples (i, j, k) with $1\leqslant j<k\leqslant 4$, $1\leqslant i\leqslant 4$ and neither $i=j$ nor $i=k$.

Apply Lemma 6 and the convexity of $\mathscr{E}_4$ twice: first $P^{(1)}:=\frac{1}{2}(P+P_1)\in\mathscr{E}_4$ and then $P^{(1)(2)}:=\frac{1}{2}(P^{(1)}+(P^{(1)})_2)\in\mathscr{E}_4$. Now $P^{(1)(2)}$ is obtained from P by deleting all terms except those in which the powers of x_1, x_2 are both odd. Thus

$$\begin{aligned}P^{(1)(2)}(\mathbf{x})&=x_1x_2x_3x_4+b_{12}x_1x_2^3+b_{21}x_2x_1^3+d_{312}x_3^2x_1x_2+d_{412}x_4^2x_1x_2\\&:=x_1x_2x_3x_4+E^{(1)(2)}(\mathbf{x}).\end{aligned}$$

We apply Theorem 2(c) to this polynomial. Evaluating $\partial E^{(1)(2)}/\partial x_3$ and $\partial E^{(1)(2)}/\partial x_4$ at $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ gives $d_{312}=d_{412}=0$. Next, computing $\partial E^{(1)(2)}/\partial x_1$ and $\partial E^{(1)(2)}/\partial x_2$ at the same point gives

$$b_{12}+3b_{21}=0; \qquad 3b_{12}+b_{21}=0.$$

Consequently $b_{12}=b_{21}=0$.

A similar argument shows that every b_{ij} and d_{ijk} is 0, and so

$$P(\mathbf{x})=x_1x_2x_3x_4+\sum_{1\leqslant i\leqslant 4} a_ix_i^4+\sum_{1\leqslant i<j\leqslant 4} c_{ij}x_i^2x_j^2=x_1x_2x_3x_4+E(\mathbf{x}).$$

The usual routine with $\partial E/\partial x_i$ evaluated at $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ for each $1 \leqslant i \leqslant 4$ gives

$$2a_1 + c_{12} + c_{13} + c_{14} = 0$$

$$2a_2 + c_{12} + c_{23} + c_{24} = 0$$

$$2a_3 + c_{13} + c_{23} + c_{34} = 0$$

$$2a_4 + c_{14} + c_{24} + c_{34} = 0.$$

A moment's thought leads to the conclusion that

$$P(\mathbf{x}) = x_1 x_2 x_3 x_4 + \sum_{1 \leqslant i < j \leqslant 4} \gamma_{ij} (x_i^2 - x_j^2)^2$$

with $\gamma_{ij} = -\frac{1}{2} c_{ij}$.

Although we cannot give a complete description of $\mathscr{E}_4$, we can say that all *small* perturbations of the Nachbin polynomial $N(\mathbf{x}) = x_1 x_2 x_3 x_4$ of the type described in Theorem 8 will be normalized extremals.

THEOREM 9. *Let* $|a| \leqslant 4^{-4}$ *and* $|b| \leqslant 4^{-4}$. *Then*

$$P(\mathbf{x}) = x_1 x_2 x_3 x_4 + a(x_1^2 - x_2^2)^2 + b(x_3^2 - x_4^2)^2$$

is a polynomial in $\mathscr{E}_4$.

COROLLARY 10. *If* $|\gamma_{ij}| \leqslant \frac{1}{3} \cdot 4^{-4}$ *for each* $1 \leqslant i < j \leqslant 4$, *then*

$$P(\mathbf{x}) = x_1 x_2 x_3 x_4 + \sum_{1 \leqslant i < j \leqslant 4} \gamma_{ij} (x_i^2 - x_j^2)^2$$

is a polynomial in $\mathscr{E}_4$.

The corollary follows from the theorem by the convexity of $\mathscr{E}_4$.

Proof of Theorem 9. The polynomial described clearly satisfies (i) and (iii) of the definition of $\mathscr{E}_4$, and $P(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = 4^{-4}$. So we just have to show that $|P(\mathbf{x})| \leqslant 4^{-4}$ whenever $|x_1| + |x_2| + |x_3| + |x_4| = 1$. In fact, since we are placing no restriction on the signs of a, b it is enough to show that $|P(\mathbf{x})| \leqslant 4^{-4}$ whenever $x_1 + x_2 + x_3 + x_4 = 1$ and $x_i \geqslant 0$ $(1 \leqslant i \leqslant 4)$.

On the boundary of this region, at least one of the x_i's is zero and so

$$\begin{aligned} |P(\mathbf{x})| = |a(x_1^2 - x_2^2)^2 + b(x_3^2 - x_4^2)^2| &\leqslant |a| \max(x_1^4, x_2^4) + |b| \max(x_3^4, x_4^4) \\ &\leqslant 4^{-4}(\max(x_1, x_2) + \max(x_3, x_4)) \leqslant 4^{-4}. \end{aligned}$$

Consequently we can focus our attention on the local extrema of $P(\mathbf{x})$ subject to the condition $x_1+x_2+x_3+x_4=1$ and $x_i>0$ $(1\leqslant i\leqslant 4)$. The Lagrange multiplier method shows that at local maxima or minima of $P(\mathbf{x})$ under the constraint $x_1+x_2+x_3+x_4=1$, there is a λ for which

$$\frac{\partial P}{\partial x_1}=x_2x_3x_4+4ax_1(x_1^2-x_2^2)=\lambda$$

$$\frac{\partial P}{\partial x_2}=x_1x_3x_4-4ax_2(x_1^2-x_2^2)=\lambda.$$

Subtracting, we get

$$(x_1-x_2)\,x_3x_4=4a(x_1-x_2)(x_1+x_2)^2.$$

A similar procedure with $\partial P/\partial x_3$ and $\partial P/\partial x_4$ gives

$$(x_3-x_4)\,x_1x_2=4b(x_3-x_4)(x_3+x_4)^2.$$

Hence there are four possible situations at a local extremum:

(1) $x_1=x_2,\ x_3=x_4$

(2) $x_1=x_2,\ x_1x_2=4b(x_3+x_4)^2$

(3) $x_3=x_4,\ x_3x_4=4a(x_1+x_2)^2$

(4) $x_3x_4=4a(x_1+x_2)^2,\ x_1x_2=4b(x_3+x_4)^2.$

Case (1). Here $P(\mathbf{x})=x_1x_2x_3x_4$ and we already know that $|P(\mathbf{x})|\leqslant 4^{-4}$ under the given conditions.

Case (2). Here

$$\begin{aligned}P(\mathbf{x})&=x_1x_2x_3x_4+b(x_3^2-x_4^2)^2\\&=4bx_3x_4(x_3+x_4)^2+b(x_3-x_4)^2\,(x_3+x_4)^2\\&=b(x_3+x_4)^4\end{aligned}$$

and so it is easy to see that $|P(\mathbf{x})|\leqslant 4^{-4}$ under the given conditions.

Case (3) is analogous to Case 2.

Case (4). At a local extremum of this type, we have $a(x_1^2-x_2^2)^2=\frac{1}{4}x_3x_4(x_1-x_2)^2$ and $b(x_3^2-x_4^2)^2=\frac{1}{4}x_1x_2(x_3-x_4)^2$, and so at these points the value of P agrees with the value of Q, where

$$\begin{aligned}Q(\mathbf{x})&:=x_1x_2x_3x_4+\tfrac{1}{4}x_1x_2(x_3-x_4)^2+\tfrac{1}{4}x_3x_4(x_1-x_2)^2\\&=\tfrac{1}{4}x_1x_2(x_3^2+x_4^2)+\tfrac{1}{4}x_3x_4(x_1^2+x_2^2)=\tfrac{1}{4}(x_1x_3+x_2x_4)(x_1x_4+x_2x_3).\end{aligned}$$

Notice that when $x_1+x_2+x_3+x_4=1$ and each $x_i>0$,

$$0<Q(\mathbf{x})\leqslant\frac{1}{4}\left[\frac{(x_1x_3+x_2x_4)+(x_1x_4+x_2x_3)}{2}\right]^2=\frac{1}{4^2}(x_1+x_2)^2(x_3+x_4)^2$$
$$\leqslant\max_{0\leqslant t\leqslant 1} t^2(1-t)^2/4^2=4^{-4}.$$

Consequently, no local extremum of P in this region has absolute value greater than 4^{-4}.

The proof is complete.

Remark. If $P(\mathbf{x})=x_1x_2x_3x_4+a(x_1^2-x_2^2)^2+b(x_3^2-x_4^2)^2$, then $P(\mathbf{e}_1)=a$ and $P(\mathbf{e}_3)=b$. Consequently $P\in\mathscr{E}_4$ if and only if $|a|\leqslant 4^{-4}$ and $|b|\leqslant 4^{-4}$.

The structure of $\mathscr{E}_m$ gets increasingly complicated as m increases. For example, using techniques similar to those in the proof of Theorem 9, it can be shown that the polynomial

$$P(\mathbf{x})=x_1\cdots x_{2k}+a(x_1^2-x_2^2)^k$$

is in $\mathscr{E}_{2k}$ when $|a|\leqslant(2k)^{-2k}$. In a more general vein, if $m=m_1+m_2$, elements of $\mathscr{E}_{m_1}$ and $\mathscr{E}_{m_2}$ can be used to produce elements of $\mathscr{E}_m$.

PROPOSITION 11. *Let $P\in\mathscr{E}_{m_1}$ and $Q\in\mathscr{E}_{m_2}$. Then the (m_1+m_2)-homogeneous polynomial $R\colon \ell_1^{m_1+m_2}\to\mathbb{R}$ given by*

$$R(x_1,...,x_{m_1+m_2})=P(x_1,...,x_{m_1})\cdot Q(x_{m_1+1},...,x_{m_2})$$

is in $\mathscr{E}_{m_1+m_2}$.

Proof. R certainly satisfies condition (i) of the definition of $\mathscr{E}_{m_1+m_2}$. To check conditions (ii) and (iii), first note that if $\sum_{1\leqslant i\leqslant m_1+m_2}|x_i|=1$ and $\sum_{1\leqslant i\leqslant m_1}|x_i|=t$, then

$$|R(x_1,...,x_{m_1+m_2})|\leqslant t^{m_1}\|P\|(1-t)^{m_2}\|Q\|.$$

Elementary calculus shows that for $0\leqslant t\leqslant 1$,

$$t^{m_1}(1-t)^{m_2}\leqslant\left(\frac{m_1}{m_1+m_2}\right)^{m_1}\left(\frac{m_2}{m_1+m_2}\right)^{m_2}$$

with the maximum being achieved at $t=m_1/(m_1+m_2)$. Plugging in the values of $\|P\|$ and $\|Q\|$, we find that

$$\|R\|\leqslant\frac{1}{(m_1+m_2)^{m_1+m_2}}.$$

Evidently $\check{R}(\mathbf{e}_1, ..., \mathbf{e}_{m_1+m_2}) = 1/(m_1+m_2)!$, and since

$$\|\check{R}\| \leqslant \frac{(m_1+m_2)^{m_1+m_2}}{(m_1+m_2)!} \|R\| \leqslant \frac{1}{(m_1+m_2)!}$$

it follows that $\|\check{R}\| = 1/(m_1+m_2)!$. So condition (iii) is satisfied, But then the above inequalities are equalities. In particular $\|R\| = (m_1+m_2)^{-(m_1+m_2)}$, and so condition (ii) also holds for R.

COROLLARY 12. *Let $m > 4$. If $|\gamma_{ij}| \leqslant (1/3)(1/4^4)$ $(1 \leqslant i < j \leqslant m)$, then*

$$P(\mathbf{x}) = \left(x_1 x_2 x_3 x_4 + \sum_{1 \leqslant i < j \leqslant 4} \gamma_{ij} (x_i^2 - x_j^2)^2 \right) x_5 \cdots x_m$$

is in $\mathscr{E}_m$.

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